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New Mathematics of Information Homotopical

Steven Awodey
CARNEGIE MELLON UNIVERSITY

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DR. STEVEN AWODEY

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14. ABSTRACT
The research funded by this award introduced a new paradigm in informatics based on a new mathematical analysis of computation. It employed a connection between Geometry, Algebra, and Logic via an interpretation of constructive type theory into homotopy theory, discovered in 2005 by the PI and his students. In addition to its intrinsic mathematical importance, this connection has resulted in a new ``geometry of computation". Powerful machine implementations of type theory in the form of proof assistants permit partial automation of reasoning in such systems; under the new homotopical interpretation, such formal reasoning can encompass abstract programming languages, constructive mathematics, and

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FINAL REPORT
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NEW MATHEMATICS OF INFORMATION: HOMOTOPICAL AND HIGHER CATEGORICAL FOUNDATIONS OF INFORMATION AND COMPUTATION

PRINCIPLE INVESTIGATOR:
Steve Awodey (Carnegie Mellon University)

PROGRAM OFFICER:
Tristan Nguyen

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Abstract

The research funded by this award introduced a new paradigm in informatics based on a new mathematical analysis of computation. It employed a connection between Geometry, Algebra, and Logic via an interpretation of constructive type theory into homotopy theory, discovered around 2005 by the PI and his students. In addition to its intrinsic mathematic importance, this connection has resulted in a new "geometry of computation". Powerful machine implementations of type theory in the form of proof assistants already permit partial automation of reasoning in such systems; under the new homotopical interpretation, such formal reasoning can encompass abstract programming languages, constructive mathematics, and large swaths of classical mathematics, including systems as powerful as ZF set theory.

The work was pursued at the Institute for Advanced Study (Princeton) in 2012–13 in a program co-organized by the PI. A group of leading logicians, computer scientists, and mathematicians developed algorithms to support the new foundations, furthering its applications to pure and applied mathematics and computation, and enhanced existing proof assistants to implement them. This work will lead to the wide-spread use of computational proof assistants, large-scale formalization of mathematics, and the creation of powerful scientific tools with impact on challenging problems of DoD interest.

1 Overview

The research funded by this award introduced and advanced a new paradigm in informatics and mathematics based on a new analysis of information and computation. It employed a newly discovered connection between Geometry, Algebra, and Logic in the form of an interpretation of constructive Martin-Löf type theory into homotopy theory. This connection was discovered around 2005 by the PI and his students. The computational tractability of the type theory and its existing machine implementations also present the exciting promise of higher-mathematical applications to the theory of computation, specifically in automated and autonomous reasoning. In addition to its intrinsic mathematic importance as a new tool in the study of homotopy and higher-dimensional category theory, this fundamental connection has opened up the possibility of a new mathematical analysis of information which may be termed the "geometry of information and computation". The use of constructive type theory as a framework for high-level programming is now well-established. On the basis of the *propositions-as-types* paradigm – also known as the *Curry-Howard correspondence* – the logical structure of data types is related to the conditions for reasoning about and computing with the data. Powerful machine implementations of constructive type theory in the form of proof assistants like Coq and Agda then make it possible to partially automate reasoning in such systems. However, under the new homotopical interpretation of type theory into simplicial sets and related abstract models of homotopy theory – which Voevodsky has dubbed the "Univalent Interpretation" [**?**] – such formal reasoning can encompass not only abstract programming languages and constructive mathematics, but also large swaths of classical mathematics, including systems as powerful as ZF set theory.

This new approach was pursued during a special year at the Institute for Advanced Study (Princeton) in 2012–13, in a program co-organized by the PI. A group of leading researchers from logic, computer science, homotopy theory, and category theory were selected from around the world to spend one year at IAS. While there they developed algorithms to support the new foundations, furthered its applications to pure and applied mathematics and computation, and enhanced existing proof assistants and developing new ones to implement the new foundations. This is expected to lead to the future wide-spread use of computational proof assistants. The resulting large-scale formalization of logic and mathematics will enable the creation of powerful scientific tools with impact on challenging problems of DoD interest.

Significant Accomplishments

The main research products are represented by the scientific books and papers written and published during the award period, and the on-line resources such as website and wiki and the repository of code, as indicated below.

1. Investigated higher-dimensional inductively defined structures and established the theory of homotopy initial algebras, published as:

   Steve Awodey, Nicola Gambino, Kristina Sojakova, "Inductive Types in Homotopy Type Theory", Logic in Computer Science, 2012.

2. Together with former PhD student Michael Warren and another researcher, established the equivalence between logical and algebraic presentations of groupoids, published as:

   Steve Awodey, Pieter Hofstra, Michael Warren, "Martin-Löf complexes", Annals of Pure and Applied Logic, 2012.

3. Investigated higher-dimensional inductive types such as the n-spheres, and initiated a new approach to the open problem of determining their computational character. Partial results reported in CMU technical report by PhD student Kristina Sojakova:

   CMU-CS-14-101R: Higher Inductive Types as Homotopy-Initial Algebras, Kristina Sojakova

4. Established and co-managed a repository of machine code for the computational implementation of logic and mathematics, including higher algebra and homotopy theory. This repository is located at:

   `https://github.com/HoTT/HoTT`

   It is under active development, and currently has over 1,600 commits, 20 contributors, and consists of many thousands of lines of code.

5. Established a webpage and blog as a web-based resource for the collection and dissemination of information, see

   `https://www.homotopytypetheory.org`

6. Organized focused sections, or spoke as invited plenary speaker, at several major international conferences in logic, mathematics, and computer science, including Association for Symbolic Logic, Logic in Com-

puter Science, Mathematical Foundations of Programming Semantics, and the American Mathematical Society.

7. Formed an on-going research group at CMU with CS Prof. Robert Harper and Math/Philosophy Prof. Jeremy Avigad, along with several PhD students and post-docs.

8. Set up an email group `homotopytypetheory@googlegroups.com`, which is now a very active and useful resource.

9. Organized a special year-long research program at the Institute for Advanced Study (Princeton) for 2012-13, focussed on grant research area. See below for further details.

10. The modified Coq system is now a fully functioning proof assistant that serves the purposes of Homotopy Type Theory and Univalent Foundations. The required modifications were planned and implemented during the IAS special year and thereafter.

11. One of the most important recent products is the comprehensive book:

    Homotopy Type Theory: Univalent Foundations of Mathematics, The Univalent Foundations Program, Institute for Advanced Study, 2013.

    This book records the main mathematical results of the IAS special year, and serves as an introduction to the subject for new-comers. It was written collaboratively by the 30 members of the special year, in a project organized and led by the PI.

12. Co-edited a special issue of the journal *Mathematical Structures in Computer Science* containing 12 articles related to the research program and authored by members during the special year.

13. Supervised or co-supervised two dissertations:

    (a) Spencer Breiner, Pure and Applied Logic, CMU: *A scheme representation of first-order logic*. Presentation of the foundations of model theory in sheaf-theoretic form on the model of Grothendieck's reformulation of hte foundations of algebraic geometry, including a definition of the notion of an elementary logical scheme.

(b) Chris Kapulkin, Mathematics, University of Pittsburgh: *The quasi-category of homotopy type theory*. Establishes that the syntactic category associated with the system of homotopy type theory gives rise to a locally cartesian closed quasi-category in the sense of Joyal. Co-supervised with Thomas Hales, University of Pittsburgh.

Also supervised one doctoral student still conducting research: Kristina Sojakova, Computer Science, CMU.

14. Wrote or co-wrote 10 scientific papers published in refereed journals or conference proceedings and publicly available. See below for details.

The IAS Special Program

The PI co-organized a year-long research program for 2012-13 at the Institute for Advanced Study, Princeton, centered on the award topic. Some 40 researchers from various institutions participated, developing a new system of foundations for mathematics based on homotopy type theory, with intrinsic computational content and well-suited to handling information. They produced a research book and a modified computer-aided proof assistant system. The program was coorganized by IAS professor Vladimir Voevodsky from the IAS school of mathematics, Steve Awodey of Carnegie Mellon University, and Thierry Coquand of Goteburg University, Sweden. See: `http://www.math.ias.edu/node/2610`.

In all, 38 full time members of the school participated in the program for at least one semester. There were also 24 short term visitors in the course of the year, as well as 6 student participants. These researchers were specialists in mathematics, logic, and computer science with diverse backgrounds ranging from homotopical algebra and category theory to theoretical and practical aspects of computational proof assistants. A typical week consisted of a full program of events including daily lectures, seminars, working groups, tutorials, and informal collaborations. For full lists of all seminars and tutorials, see:

`http://uf-ias-2012.wikispaces.com/Seminar`

`http://uf-ias-2012.wikispaces.com/Tutorials`

Several important products were created during the program, one of which is a wiki which is an important resource for researchers in the field. Itcan be viewed at: `https://www.UF-IAS-2012.wikispaces.com`

Early in the year, a working group devoted to the Coq system was organized and focused on modifying the Coq proof assistant to be more useful for Univalent Foundations. By the end of the first term, a working system was in place for use in the second term, and work on a next generation proof assistant was begun. Another working group was devoted to developing a systematic, informal style of type theory. This project soon evolved into the large-scale project of writing of a book, in which the basics of Univalent Foundations and the results of the special year are developed in an exemplary informal style. This book, which is availble free online, is the result of a remarkable collaborative effort and should serve as a useful resource for disseminating the results of the special year. A third working group was devoted to the development of basic homotopy theory in Univalent Foundations, including new type theoretic proofs and their formalization in the Coq system. Many homotopy groups of spheres were calculated and their proofs formalized, as were other classic results of homotopy theory.

Impact

The research supported by this award has already had a surprizingly great impact on several scientific communities, especially considering the relatively short time period involved. Homotopy type theory is now established as an active branch of study within several research communities, including type theory in logic, homotopy theory in pure mathematics, and computational proof assistants and program verification in computer science. There have been plenary lectures and research sessions at major international conferences (such as the Association for Symbolic Logic, The American Mathematical Society, and Logic in Computer Science), long-term thematic research programs at several leading institutions (such as the Institute for Advanced Study in Princeton and the Institut Henri Poincaré in Paris); and research and survey articles and special issues (several of them invited) published by the PI and his collaborators in many leading scientific journals (such as the *Notices of the American Mathematical Society* and *Mathematical Structures in Computer Science*). The research community of individuals involved in current research of the various aspects of the program – in Mathematics, Logic, and Computer Science – now comprises dozens of individuals at scores of institutions worldwide. Key issues are being isolated and rapid advances are being made. The new discipline shows every indication that it will continue to grow and to contribute significant advances in the near future and long-term.

Publications

Authored or coauthored by the PI during the award period:

1. *Homotopy Type Theory: Univalent Foundations of Mathematics*, The Univalent Foundations Program, Institute for Advanced Study, 2013. Available at: `http://homotopytypetheory.org/book`

2. Natural models of homotopy type theory, on the arXiv as `arXiv:1406.3219`.

3. Topos Semantics for Higher-Order Modal Logic, (with K. Kishida and H.-C. Kotzsch). To appear in *Logique et Analyse*.

4. Voevodsky's Univalence Axiom in Homotopy Type Theory, (with Á. Pelayo and M. A. Warren). Notices of the American Mathematical Society, 2013.

5. Structuralism, Invariance, and Univalence, *Philosophia Mathematica*, 2013.

6. Homotopy Type Theory and the Large-Scale Formalization of Mathematics, (with T. Coquand), *The Institute Letter, Institute for Advanced Study*, 2013.

7. First-Order Logical-Duality (with H. Forssell), *Annals of Pure and Applied Logic*, 2013.

8. Martin-Löf Complexes (with P. Hofstra and M. Warren). *Annals of Pure and Applied Logic*, 2013.

9. Relating Topos Theory and Set Theory via Categories of Classes, (with C. Butz, A. Simpson, T. Streicher). *Annals of Pure and Applied Logic*, 2013.

10. Inductive Types in Homotopy Type Theory, (with N. Gambino and K. Sojakova). Logic in Computer Science (LICS) 2012.

11. Type Theory and Homotopy. In *Essays on the Foundations of Mathematics in Honor of Per Martin-Löf*, Edited by Peter Dybjer et al., Springer 2012.